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COMPLETENESS OF LORENTZ MANIFOLDS OF CONSTANT CURVATURE ADMITTING KILLING VECTOR FIELDS

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Dedicated to Professor Akio Hattori on his sixtieth birthday

Introduction

A Lorentz manifold M of dimension n is a smooth manifold together with a Lorentz metric g. A Lorentz metric g on M is a smooth field $\{g_x\}_{x\in M}$ of nondegenerate symmetric bilinear forms g_x of type $(1, n-1)$ on the tangent space T_xM. Namely let $\mathbf{R}^{1,n-1}$ denote the real vector space of dimension n equipped with the bilinear form

$$Q(x, y) = -x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

A nondegenerate symmetric bilinear form g_x is of type $(1, n-1)$ if the pair (T_xM, g_x) is isometric to $(\mathbf{R}^{1,n-1}, Q)$ (cf. [31], [34]).

A pseudo-Riemannian manifold has a unique connection (Levi-Cività connection) on its frame bundle. Henceforth geodesics, curvature, completeness refer to the Levi-Cività connection. It is notorious that compactness does not necessarily imply completeness in *pseudo-Riemannian geometry.* In this paper we consider this problem for Lorentz manifolds of constant curvature which admit Killing vector fields of certain type. This leads to some precise classification results.

Theorem A. *Let M be a compact Lorentz manifold of constant curvature k. Suppose that M admits a timelike Killing vector field. Then M is complete, $k \leq 0$ and the following hold:*

(1) *M is affinely diffeomorphic to a euclidean space form with nonzero first Betti number if $k = 0$;*

(2) *some finite covering of M is a circle bundle over a negatively curved manifold if k is a negative constant.*

This will be proved in Corollary 3.2, Theorem 2.15, and Theorem 2.17. A compact Lorentz manifold of $k = 0$ is called a Lorentz flat manifold. It is known that a compact Lorentz flat manifold is complete by the result

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