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THE LIMITING ETA INVARIANTS OF COLLAPSED THREE-MANIFOLDS

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In this paper, we study the limiting eta invariants of collapsed Riemannian manifolds. These invariants were defined and previously studied in [9]. In particular, we prove a conjecture of Cheeger and Gromov which asserts their rationality in the three-dimensional case, provided that the collapse has bounded covering geometry.

0. Introduction

Let M be an n-dimensional complete Riemannian manifold with sectional curvature bounded in absolute value, say $|K| \leq 1$. Let $\alpha(g)$ denote one of the following geometric quantities associated to g: the diameter of M, the supremum of injectivity radii at all points of M, or the volume of M. Roughly speaking, M is said to be *sufficiently* $\alpha(g)$*-collapsed*, if $\alpha(g)$ is smaller than a sufficiently small constant depending only on dimension of M. M is said to admit an $\alpha(g)$*-collapse*, if there exists a family of metrics $\{g_\delta\}$ on M, $0 < \delta \leq 1$, such that the sequence $\{\alpha(g_\delta)\}$ converges to zero as $\delta \to 0$ (here we assume the sectional curvatures of all g_δ are bounded in absolute value by one).

The basic questions about the interplay between the collapsing geometry and the topology of M are the following:

(1) What kind of structures and invariants can be attached to a sufficiently α-collapsed metric or to an α-collapse ?

(2) Does a sufficiently α-collapsed metric imply the existence of an α-collapse ?

Starting with [21], there has been considerable progress on the above questions; for instance, Gromov's theorem of almost flat manifolds [21], i.e., manifolds whose diameter is sufficiently collapsed, the F-structure theory for sufficiently collapsed injectivity radii [10], [11], the bundle structure theorems and their applications for manifolds which collapse to a

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