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ALGEBRAIC GEOMETRIC INTERPRETATION OF DONALDSON'S POLYNOMIAL INVARIANTS

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0. Introduction

The purpose of this paper is to understand Donaldson's polynomial invariants of four-manifolds in the context of algebraic geometry. In particular, we explore, among other things, the possibility of defining Donaldson's polynomial invariants of algebraic surfaces by relying on the intersection theory of algebraic varieties.

Let X be any smooth algebraic surface and let H be a very ample line bundle on X with $\mathbf{g}$ its Hodge metric. For any $d > 0$, there is a unique SU(2) bundle E over X of second Chern class d. We denote by $\mathscr{N}_d(\mathbf{g})$ the space of gauge equivalent classes of irreducible anti-self-dual (ASD) (with respect to $\mathbf{g}$) SU(2) connections on E. According to Uhlenbeck's weak compactness theorem, there is a canonical compactification of $\mathscr{N}_d(\mathbf{g})$ [2]. Let $\overline{\mathscr{N}}_d(\mathbf{g})$ be such a compactification. On the other hand, any irreducible ASD connection on E induces a holomorphic structure on E, which turns out to be μ-stable with respect to the divisor H [1]. Thus, $\mathscr{N}_d(\mathbf{g})$ can be identified with a subset of the moduli scheme $\mathfrak{M}_d(H)$ of rank two H-semistable sheaves F with $\det F = \mathscr{O}$ and $c_2(F) = d$. $\mathfrak{M}_d(H)$ is projective, thus is compact [8]. In this sense, $\mathfrak{M}_d(H)$ is another compactification of the space $\mathscr{N}_d(\mathbf{g})$. It is both interesting and important to understand the relation between Uhlenbeck's compactification $\overline{\mathscr{N}}_d(\mathbf{g})$ and Gieseker's compactification $\mathfrak{M}_d(H)$. Based on Uhlenbeck's compactification $\overline{\mathscr{N}}_d(\mathbf{g})$, Donaldson introduced a series of polynomials $\{q_k\}$ of the four-manifold X. The polynomials are defined by calculating the self-intersection numbers of proper subsets of $\overline{\mathscr{N}}_d(\mathbf{g})$ when $\mathbf{g}$ is a generic Riemannian metric. In the case where the manifold X is simply connected and $b_+^2(X)$ is odd and strictly larger than 1, he showed that these numbers are well defined and are indeed invariants of the smooth structure of the four-manifold X [3].

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