J. DIFFERENTIAL GEOMETRY
37 (1993) 375-396

DURFEE CONJECTURE AND COORDINATE FREE CHARACTERIZATION OF HOMOGENEOUS SINGULARITIES

YI-JING XU & STEPHEN S.-T. YAU

0. Introduction

This work is a natural continuation of our previous work [14].

The motivation of our work is to solve the Durfee conjecture. Let $f: (\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$ be the germ of a complex analytic function with an isolated critical point at the origin. For $\varepsilon > 0$ suitably small and δ yet smaller, the space $V' = f^{-1}(\delta) \cap D_\varepsilon$ (where D_ε denotes the closed disk of radius ε about 0) is a real oriented four-manifold with boundary whose diffeomorphism type depends only on f. It has been proved that V' has the homotopy type of a wedge of two-spheres; the number μ of two-spheres is precisely $\dim \mathbf{C}\{x, y, z\}/(f_x, f_y, f_z)$. Let $\pi: (M, A) \to (V, 0)$ be a resolution of $V = \{(x, y, z) : f(x, y, z) = 0\}$ with exceptional set $A = \pi^{-1}(0)$. The geometric genus p_g of the singularity V is the dimension of $H^1(M, \mathscr{O})$. Let $\chi(A)$ be the topological Euler characteristic of A, and K^2 be the self-intersection number of the canonical divisor on M. Laufer's formula (cf. [5]) says that

$$1 + \mu = \chi(A) + K^2 + 12p_g.$$

However the formula does not provide direct comparison between μ and p_g, which are two important numerical measures of the complexity of the singularity. In 1978, Durfee [2] made the following spectacular conjecture which has remained open ever since.

Durfee conjecture. Let σ be the signature of the Milnor fiber V' above. Then

(1) $\sigma \leq 0$,

(2) $6p_g \leq \mu$ with equality only when $\mu = 0$.

In this paper we prove the Durfee conjecture in the weighted homogeneous case. In fact we show that the conjecture itself is not sharp. More precisely, we have the following theorem.

Received November 19, 1990 and, in revised form, August 1, 1991. The authors' research was partially supported by a National Science Foundation Grant.