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A NUMERICAL CRITERION FOR VERY AMPLE LINE BUNDLES

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Abstract

Let X be a projective algebraic manifold of dimension n and let L be an ample line bundle over X. We give a numerical criterion ensuring that the adjoint bundle $K_X + L$ is very ample. The sufficient conditions are expressed in terms of lower bounds for the intersection numbers $L^p \cdot Y$ over subvarieties Y of X. In the case of surfaces, our criterion gives universal bounds and is only slightly weaker than I. Reider's criterion. When $\dim X \geq 3$ and $\operatorname{codim} Y \geq 2$, the lower bounds for $L^p \cdot Y$ involve a numerical constant which depends on the geometry of X. By means of an iteration process, it is finally shown that $2K_X + mL$ is very ample for $m \geq 12n^n$. Our approach is mostly analytic and based on a combination of Hörmander's L^2 estimates for the operator $\overline{\partial}$, Lelong number theory and the Aubin-Calabi-Yau theorem.

1. Introduction

Let L be a holomorphic line bundle over a projective algebraic manifold X of dimension n. We denote the canonical line bundle of X by K_X and use an additive notation for the group $\operatorname{Pic}(X) = H^1(X, \mathcal{O}^*)$. The original motivation of this work was to study the following tantalizing conjecture of Fujita [23]: If $L \in \operatorname{Pic}(X)$ is ample, then $K_X + (n+2)L$ is very ample; the constant $n+2$ would then be optimal since $K_X + (n+1)L = \mathcal{O}_X$ is not very ample when $X = \mathbf{P}^n$ and $L = \mathcal{O}(1)$. Although such a sharp result seems at present out of reach, a consequence of our results will be that $2K_X + mL$ is always very ample for L ample and m larger than some universal constant depending only on n.

Questions of this sort play a very important role in the classification theory of projective varieties. In his pioneering work [9], Bombieri proved the existence of pluricanonical embeddings of low degree for surfaces of general type. More recently, for an ample line bundle L over an algebraic surface S, I. Reider [39] obtained a sharp numerical criterion ensuring that the adjoint line bundle $K_X + L$ is very ample; in particular,

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