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THE HARNACK ESTIMATE FOR THE RICCI FLOW

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1. The result

1.1. Main Theorem. *Let g_{ij} be a complete solution with bounded curvature to the Ricci flow*

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$$

on a manifold M for t in some time interval $0 < t < T$ and suppose g_{ij} has a weakly positive curvature operator, so that

$$R_{ijkl} U_{ij} U_{kl} \geq 0$$

for all two-forms U_{ij}. Let

$$P_{ijk} = D_i R_{jk} - D_j R_{ik}$$

and let

$$M_{ij} = \Delta R_{ij} - \frac{1}{2} D_i D_j R + 2R_{ikjl} R_{kl} - R_{ik} R_{jk} + \frac{1}{2t} R_{ij} .$$

Then for any one-form W_i and any two-form U_{ij} we have

$$M_{ij} W_i W_j + 2P_{ijk} U_{ij} W_k + R_{ijkl} U_{ij} U_{kl} \geq 0 .$$

1.2. Corollary. *For any one-form V_i we have*

$$\frac{\partial R}{\partial t} + \frac{R}{t} + 2D_i R \cdot V_i + 2R_{ij} V_i V_j \geq 0 .$$

The corollary follows immediately by taking

$$U_{ij} = \tfrac{1}{2}(V_i W_j - V_j W_i)$$

and tracing over W_i.

The existence of inequalities on the second derivatives of solutions of parabolic equations was first noted by Peter Li and S.-T. Yau [12] for the scalar heat flow on a Riemannian manifold. The author has observed a similar phenomenon for the matrix of second derivatives in the scalar heat

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