AN EXOTIC MENAGERIE

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0. Introduction

Among the most peculiar examples in topology are the *exotic* $\mathbb{R}^4$'s. These are smooth 4-manifolds which are homeomorphic to a Euclidean 4-space $\mathbb{R}^4$ but not diffeomorphic to it. Exotic $\mathbb{R}^4$'s represent a phenomenon which is fundamentally unique to dimension 4, since it is a central result of smoothing theory that, for $n \neq 4$, any smooth manifold homeomorphic to $\mathbb{R}^n$ must be diffeomorphic to it. Consequently, exotic $\mathbb{R}^4$'s provide counterexamples to various basic conjectures about the extension of high-dimensional topology to dimension 4. For example, in contrast with high dimensions, exotic smooth structures on 4-manifolds cannot be adequately analyzed via an obstruction theory (since $\mathbb{R}^4$ is contractible).

There are two main approaches to constructing and distinguishing exotic $\mathbb{R}^4$'s, and these yield manifolds with rather different properties—exotic $\mathbb{R}^4$'s of one type are much "larger" than those of the other type. Both constructions rely on work of Freedman and Donaldson, and the roots of both constructions can be traced to work of Casson [4]. We will discuss these constructions in chronological order.

The original exotic $\mathbb{R}^4$ resulted from an observation of Freedman, who noted that in the presence of his own work [8] and Donaldson's nonexistence theorem [6], a certain construction of Casson yielded a manifold homeomorphic to $\mathbb{R}^4$, whose end was not diffeomorphic to $S^3 \times \mathbb{R}$. In fact, it was easily seen that this exotic $\mathbb{R}^4$ contained a compact, codimension-zero submanifold which could not be smoothly embedded in S^4. Subsequently, it was shown that more than one such example existed [12]—in fact, infinitely many [13]. Then Taubes [22] solved a problem in gauge theory posed by Freedman, which yielded an uncountable family $\{R_t | t \in (0, 1)\}$ for which R_t embedded in $R_{t'}$ if and only if $t \leq t'$. In [13], this was extended to a two-parameter family $\{R_{s,t} | s, t \in (0, 1)\}$ with natural