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HODGE THEORY AND THE HILBERT SCHEME

ZIV RAN

In Mori's "bend and break" method [6], [7], on which his theory of extremal rays is based, a key technical role is played by a fundamental estimate, due to Grothendieck, on the dimension of the Hilbert scheme of curves in an algebraic manifold [4]. Specifically, and more generally, if X is an algebraic (or complex) manifold and $Y \subset X$ is a submanifold with normal bundle N, then Grothendieck's estimate states that any component $\mathscr{H}$ of the Hilbert scheme (or Douady space) $\mathscr{H}ilb_X$ containing $\{Y\}$ satisfies

$$(1) \qquad \dim \mathscr{H} \geq h^0(N) - h^1(N).$$

In view of the fundamental, and very general, nature of the estimate (1), one naturally wonders whether it might be possible to improve it in some interesting special cases. One such improvement of a Hodge-theoretic nature is due to S. Bloch [2], generalizing some earlier work by Kodaira-Spencer in the codimension-1 case: Bloch defines a certain map

$$\pi \colon H^1(N) \to H^{p+1}(\Omega_X^{p-1}), \qquad p = \operatorname{codim}(Y, X),$$

which he names the *semiregularity* map, and proves that if π is injective, then $\mathscr{H}ilb_X$ is in fact *smooth* at $\{Y\}$, so that the estimate (1) may be improved to

$$(2) \qquad \dim \mathscr{H} = h^0(N)$$

(as is well known, $h^0(N)$ is the embedding dimension at $\{Y\}$ of $\mathscr{H}ilb_X$, hence $\dim \mathscr{H} \leq h^0(N)$ always holds, with equality iff $\mathscr{H}ilb_X$ is smooth at $\{Y\}$).

Now as Bloch's semiregularity map π can rarely be injective, his dimension estimate (2) is not very useful as it stands. However, there is a natural generalization of (2) which seems a priori quite plausible (as well as more useful): namely, the estimate

$$(3) \qquad \dim \mathscr{H} \geq h^0(N) - h^1(N) + \dim \operatorname{im}(\pi).$$

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