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UPPER BOUNDS FOR EIGENVALUES OF CONFORMAL METRICS

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0. Introduction

In this paper we shall study upper bounds for eigenvalues of the Laplacian. In the case of manifolds with boundary we will consider the Neumann eigenvalues μ_k. (We denote Dirichlet eigenvalues by λ_k.) A famous theorem of A. Weyl asserts that given a domain $\Omega \subset \mathbf{R}^n$ with finite volume V, μ_k and λ_k have asymptotic values as $k \to \infty$, given by $C_n(k/V)^{2/n}$ [13]. Here $C_n \equiv 4\pi^2\omega_n^{-2/n}$, ω_n is the volume of the unit ball in $\mathbf{R}^n$, and to say that two sequences are asymptotic means that their successive ratios approach 1. It is well known that this asymptotic formula actually holds for any compact Riemannian manifold with boundary. (See e.g. [1].) Of course, the rate at which the eigenvalues become asymptotic to $C_n(k/V)^{2/n}$ depends on the geometry of the domain or manifold one is considering.

G. Polya proved that for certain "tiling domains" in $\mathbf{R}^2$ the asymptotic formula is actually an estimate below for all the Dirichlet eigenvalues, and an estimate above for all the Neumann eigenvalues [8]. In the same work (and earlier, [7]) he conjectured that these upper and lower bounds should hold for Neumann and Dirichlet eigenvalues on general domains. More precisely, if we define $\mu_1 = 0$ to correspond to the constant function, then Polya conjectured (in the case $n = 2$) that for any finite volume $\Omega \subset \mathbf{R}^n$ we have the estimates

$$\lambda_k \geq C_n \left(\frac{k}{V}\right)^{2/n}, \tag{0.1a}$$

$$\mu_k \leq C_n \left(\frac{k-1}{V}\right)^{2/n}. \tag{0.1b}$$

The generality in which such a theorem might be true is not understood. In a survey article S. T. Yau generalizes the conjecture and asks for conditions under which such upper and lower estimates can hold for two-dimensional

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