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QUASI-SPHERICAL METRICS AND PRESCRIBED SCALAR CURVATURE

ROBERT BARTNIK

Abstract

We describe a construction for metrics of prescribed scalar curvature on $S^2 \times \mathbf{R}$, based on a *quasi-spherical* coordinate condition. The construction uses two arbitrary functions and requires the solution of a semilinear parabolic equation on S^2, with the arbitrary functions and the scalar curvature appearing as source terms. We obtain existence results for this equation under various geometrically natural boundary conditions, and thereby construct some 3-metrics of interest in general relativity.

1. Introduction

Riemannian 3-manifolds with prescribed scalar curvature arise naturally in general relativity as spacelike hypersurfaces in the underlying spacetime. If $g = (g_{ij})$, $i, j = 1, \cdots, 3$, is the induced (Riemannian) metric on the spacelike hypersurface M, then the scalar curvature $R(g)$ is determined by the extrinsic curvature (second fundamental form) K_{ij} and the spacetime energy-momentum tensor $T_{\alpha\beta}$, via the Gauss-Codazzi and Einstein equations:

$$16\pi T(e_0, e_0) = R(g) - \|K\|^2 + (\operatorname{tr}_g K)^2, \tag{1.1}$$

where e_0 is the (future) timelike unit normal of the hypersurface M, $\|K\|^2 = g^{ik}g^{jl}K_{ij}K_{kl}$, $\operatorname{tr}_g K = g^{ij}K_{ij}$, and the Einstein equations are $G_{\alpha\beta} := \operatorname{Ric}_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = 8\pi T_{\alpha\beta}$. The main situation of physical interest is where $R(g) \geq 0$—for example, if M is totally geodesic ($K_{ij} = 0$) and the spacetime is vacuum ($T_{\alpha\beta} = 0$), then $R(g) = 0$, and more generally if M is a *maximal* hypersurface ($\operatorname{tr}_g K = 0$) and the spacetime satisfies the *weak energy condition* [18], then $T(e_0, e_0) \geq 0$ and thus $R(g) \geq 0$. Provided M is suitably constrained (for example, by the maximal hypersurface condition), the metric structure of (M, g) reflects that of the ambient spacetime, and therefore it is important to understand this structure.

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