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A CHARACTERIZATION OF TEICHMÜLLER DIFFERENTIALS

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1. Introduction

1.1. A quasiconformal affine mapping Tz has the expression

$$Tz = A(z + ke^{2i\theta}\bar{z}) = Ae^{i\theta}([e^{-i\theta}z] + k\overline{[e^{-i\theta}z]}),$$

where $0 < k < 1$ if T is not conformal. From this expression we conclude that

(1) there is a unique rectangular coordinate system whose image is rectangular, and

(2) in terms of this pair of systems, T can be expressed as a pure stretch: writing $w = u + iv$, depending on the scaling, and $K = (1+k)/(1-k)$, we have

$$Tw = \begin{cases} Ku + iv & \text{(height preserving)}, \\ \sqrt{K}u + iv/\sqrt{K} & \text{(area preserving)}. \end{cases}$$

Let $g: R \to S$ be an orientation preserving homeomorphism between the compact or finitely punctured compact Riemann surfaces R and S of hyperbolic type. Starting with the assumption that a condition analogous to property (1) above holds, we will establish that the analogue to property (2) follows; that is, we will establish by an explicit construction that there is a Teichmüller mapping in the homotopy class of g. In §9 we will show that the unique axis theorem for hyperbolic (pseudo-Anosov) elements of the Teichmüller modular group is a consequence.

In the last part of the paper, it will be shown that the analogue to property (1) in fact holds. This leads to a geometric proof of the Teichmüller mapping theorem (§10).

Our first main theorem is related, via the theory of measured foliations, to the following result of Masur [11] as completed in Gardiner-Masur [3]. Namely, an equivalence class of transverse measured foliations on a surface determines a unique complex structure in terms of which it is realized

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