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THE TOPOLOGY OF THE SPACE OF STABLE BUNDLES ON A COMPACT RIEMANN SURFACE

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Abstract

We develop a singular Morse theory for the Yang-Mills functional on the space of holomorphic structures on a bundle over a compact Riemann surface. We also examine the relation with the algebraic methods.

1. Introduction

The cohomology of the moduli space of semistable bundles over a curve was computed by algebraic methods by Harder and Narasimhan [13] and by transcendental methods by Atiyah and Bott [1]. The main idea behind the second approach was to employ techniques from differential and symplectic geometry. Roughly speaking, the approach of Atiyah and Bott is based on the observation that for many moduli problems, including that of holomorphic bundles, the algebraic geometric notion of stability is related to the Morse theory of certain associated functionals.

F. Kirwan pursued this observation further, by examining a very broad class of group actions on nonsingular projective varieties and symplectic manifolds, and obtained inductive formulas for computing the equivariant cohomology of the set of semistable points of these actions ([16], [17]). She also indicated explicitly how Morse theory is related to the ideas of geometric invariant theory and geometric quantization ([22], [11]). Results in the same direction were also obtained independently by L. Ness [24].

The situation studied by Atiyah and Bott differs from Kirwan's mainly because they are dealing with an infinite-dimensional problem. The space of holomorphic structures on a fixed smooth bundle E over a Riemann surface M is an infinite-dimensional manifold $\mathfrak{B}$. The group $\mathfrak{g}^{\mathbb{C}}$ of complex gauge transformations is an infinite-dimensional Lie group acting on $\mathfrak{B}$ and $[\mathfrak{B}] = \mathfrak{B}/\mathfrak{g}^{\mathbb{C}}$ parametrizes the space of holomorphic bundles over M of fixed topological type. Atiyah and Bott define a stratification $\{\mathfrak{B}_\mu\}$

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