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ON THE ALGEBRAIC STRUCTURE OF TWISTOR SPACES

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Introduction

The twistor space associated to a compact self-dual 4-manifold is a compact complex 3-fold whose complex structure is determined by the self-dual conformal structure of the 4-manifold. The most characteristic property of a twistor space is that it is foliated by a four-real-parameter family of rational curves with normal bundle isomorphic to that of a line in the complex projective 3-space; indeed, the leaf-space of this foliation is precisely the associated self-dual 4-manifold [2]. The simplest example of a compact nonflat self-dual 4-manifold is the Euclidean 4-sphere; the corresponding twistor space is the complex projective 3-space. A second well-known example is the full-flag space of $\mathbf{C}^3$ as the twistor space associated to the complex projective plane $\mathbf{P}^2$ equipped with the Fubini-Study metric. As shown by Hitchin [10], the preceding two twistor spaces are the only Kählerian twistor spaces, and one might be tempted to believe that methods of algebraic geometry would therefore be of no avail in the study of self-dual manifolds. However, there exist other twistor spaces that are bimeromorphic to algebraic varieties, i.e., Moishezon spaces. The first such examples of this type were described in [18], and correspond to self-dual metrics on the connected-sum of two complex projective planes $\mathbf{P}^2 \# \mathbf{P}^2$. There is in fact a 1-parameter moduli space of such metrics, and each of the corresponding twistor spaces is a small resolution of the intersection of two quadrics in $\mathbf{P}^5$ with four ordinary double points.

At this point, one might ask whether one can find *other* Moishezon twistor spaces. It turns out ([19], [4]) that the 4-manifold associated with such a twistor space must be homeomorphic to an iterated connected-sum $\tau\mathbf{P}^2 := \mathbf{P}^2 \# \cdots \# \mathbf{P}^2$ of τ copies of the complex projective plane and the self-dual conformal class contains a metric of positive scalar curvature. A most encouraging sign was therefore given by the result of Donald-

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