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UNIFORMLY ELLIPTIC OPERATORS ON RIEMANNIAN MANIFOLDS

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Abstract

Given a Riemannian manifold (M, g), we study the solutions of heat equations associated with second order differential operators in divergence form that are uniformly elliptic with respect to g. Typical examples of such operators are the Laplace operators of Riemannian structures which are quasi-isometric to g. We first prove some Poincaré and Sobolev inequalities on geodesic balls. Then we use Moser's iteration to obtain Harnack inequalities. Gaussian estimates, uniqueness theorems, and other applications are also discussed. These results involve local or global lower bound hypotheses on the Ricci curvature of g. Some of them are new even when applied to the Laplace operator of (M, g).

1. Introduction

Let (M, g) be a complete Riemannian manifold. In this paper, we study some second-order differential operators which are to g what uniformly elliptic operators in divergence form are to the Euclidean metric on $\mathbb{R}^n$. The simplest example of such an operator is the Laplace-Beltrami operator Δ associated with g. Under lower bound hypotheses on the Ricci curvature, Li and Yau have obtained definitive results concerning Δ and the corresponding heat equation. By rather elementary means, they proved in [28] a remarkable gradient estimate which implies parabolic Harnack inequalities as well as upper and lower Gaussian bounds for the kernel of the heat flow semigroup $e^{-t\Delta}$. In fact, the parabolic gradient estimate of Li-Yau is a generalization of previous elliptic results by Yau [43] and Cheng-Yau [10]. One important aspect of these works is that they give global estimates when the Ricci curvature is nonnegative on M. This is well illustrated by a well-known result of Yau who proved in [43] that any manifold (M, g) with nonnegative Ricci curvature has the strong Liouville property (i.e., any positive harmonic function on (M, g) is constant).

Let us now introduce a typical example of the operators studied in this paper. Let $\tilde{\Delta}$ be the Laplacian associated with another metric $\tilde{g}$ on M

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