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MINIMAL SUBMANIFOLDS DEFINED BY FIRST-ORDER SYSTEMS OF PDE

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Abstract

We study first-order PDE systems implying the second-order system for minimal submanifolds of a Euclidean n-space $\mathbb{R}^n$. We approach the problem geometrically by studying subsets Σ of the Grassmannian which we call *m-subsets*, where we define Σ to be an m-subset if all submanifolds of $\mathbf{R}^n$, whose Gauss map's image is contained in Σ, are automatically minimal. m-subsets generalize the faces of calibrations studied by Harvey and Lawson. We also study linear first-order systems implying Laplace's equation, the infinitesimal version of the m-subset problem. Results include new examples of classes of minimal submanifolds admitting 'Weierstrass type' presentations in terms of holomorphic data; dimension restriction and rigidity theorems for m-subsets that extend to faces of calibrations; and showing certain codimension-two minimal submanifolds of $\mathbf{R}^n$ are stable using a nonconstant coefficient calibration argument.

Introduction

PDE. It was first observed by Riemann that one could obtain two solutions to Laplace's second-order equation

$$u_{xx} + u_{yy} = 0$$

by solving the first-order Cauchy-Riemann system

$$u_x = v_y, \qquad u_y = -v_x.$$

Since then, others have studied systems of second-order PDE having related first-order systems which imply the second-order system. By 'system A implies system B,' we mean solutions of A are automatically solutions of B.

A nonlinear example of this phenomena occurs in Yang-Mills theory in four dimensions. The vector bundles with self-dual connections (a first-order system of PDE) satisfy the (second-order) Euler-Lagrange equations

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