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WEIL-PETERSSON VOLUMES

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Abstract

An explicit method of integration of top-dimensional differential forms over the moduli spaces of punctured Riemann surfaces is presented. This method is applied to the computation of Weil-Petersson volumes of moduli spaces, and we find that the volume for the twice-punctured torus is $\pi^4/8$ and that, for large g, the volume for the once-punctured surface of genus g is at least $g^{-2}c^{-2g}(2g)!$, where $c < .15$ is a constant independent of g. Our methods depend upon a certain bundle (introduced earlier) over the classical Teichmüller space of punctured surfaces, and some of our computations rely on standard techniques from quantum field theory.

1. Introduction

In [11] and [13] we proposed an explicit method of integration of top-dimensional differential forms over the moduli space $\mathscr{M}_g^s$ of the surface F_g^s of genus g with $s > 0$ punctures, and one purpose of this paper is to present a complete exposition of this integration scheme. We also gave in [11] an expression of the Weil-Petersson Kähler two-form in coordinates reasonably well suited to our method of integration and put forward the computation of the various Weil-Petersson volumes μ_g^s of $\mathscr{M}_g^s$ as test cases for the utility of our techniques.

We have had some success on these test cases, and this is also reported herein. Specifically, we have computed that the Weil-Petersson volume μ_1^2 is $\pi^4/8$. Furthermore, we derive an asymptotic expression for μ_g^1 to the effect that, for large g, $\mu_g^1 > g^{-2}c^{-2g}(2g)!$, where $c < .15$ is a constant independent of g. This latter estimate has found physical significance in [15] and agrees with predications from two-dimensional quantum gravity (see [18]).

Little is known beyond these new results about Weil-Petersson volumes. (*Added in proof*: There has been much progress on this recently; see [19]

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