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ON THE STRUCTURE OF ALMOST NONNEGATIVELY CURVED MANIFOLDS

JYH-YANG WU

1. Introduction

In this paper we shall say that a compact Riemannian manifold M is almost nonnegatively curved if the sectional curvature, K_M, and the diameter, $d(M)$, of M satisfy

$$K_M d(M)^2 > -\varepsilon$$

for some small positive number ε. In [6] and [18], Kukaya and Yamaguchi studied the fundamental groups of almost nonnegatively curved manifolds. Their results assert that there is a positive number ε_n depending only on the dimension n such that if a closed n-manifold satisfies $K_M (M)^2 > -\varepsilon_n$, then its fundamental group, $\pi_1(M)$, is almost nilpotent, i.e., $\pi_1(M)$ contains a nilpotent subgroup of finite index. Moreover, M fibers over a $b_1(M)$-torus, where $b_1(M)$ denotes the first Betti number of M.

It is proved in [3] that any closed n-manifold M with nonnegative sectional curvature is, up to a finite cover, diffeomorphic to a direct product $N \times T^k$, where N is a simply-connected smooth $(n-k)$-manifold and T^k is a k-torus. Hence, it is natural to ask whether or not this theorem still holds for almost nonnegatively curved manifolds. In general, this is not true. This can be seen from the examples: almost flat n-manifolds with fundamental groups of polynomial growth with degree $\geq n+1$. It is therefore clear that one needs additional assumptions to extend Cheeger-Gromoll's theorem to the almost nonnegatively curved manifolds.

In [15], Shen and Wei considered a lower bound on the injectivity radius, $i(M)$, of M and obtained the following theorem:

Theorem (Shen, Wei). *Given n, $D > 0$ and $i_0 > 0$, there is a positive number $\varepsilon^* = \varepsilon^*(n, D, i_0)$ depending only on n, D, and i_0 such that if M is a compact Riemannian n-manifold satisfying*

$$d(M) \leq D, \quad i(M) \geq i_0, \quad K_M > -\varepsilon^*,$$

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