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SHORTENING SPACE CURVES AND FLOW THROUGH SINGULARITIES

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Abstract

When a closed curve immersed in the plane evolves by its curvature vector, singularities can form before the curve shrinks to a point. We show how to use the curvature flow on space curves to define a natural continuation of the planar solution for all time.

0. Introduction

When a simple closed curve in the plane evolves by the curvature flow, it shrinks to a point in finite time, becoming round in the limit ([4] [5]). When the curve is not simple, however, singularities can form in finite time as loops pinch off to form cusps. The classical machinery for short-time existence of solutions to the curvature flow breaks down when the curvature becomes unbounded. This is not to say that it cannot be continued. In [2], Angenent shows that the singular curves are nice enough that, with some possible trimming, they may be used as initial data for the curve shortening flow. Solutions after the singularity have fewer self-intersections than before.

About ten years ago, Calabi suggested a method for flowing through planar singularities using space curves. The idea is to take a family Γ of embedded space curves limiting on the immersed plane curve, and then define a flow through the singularity as the limit of the flows in Γ.

Several points must be checked:

(1) The space curves must be non-singular for longer than the planar curve.

(2) The space curves must converge to a planar curve at later times.

(3) The limit planar curve should be independent of Γ.

Definition 0.1. A *ramp* is a space curve which steadily gains height, that is, its tangent vector has positive vertical component at all points.

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