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ON THE NODAL LINE OF THE SECOND EIGENFUNCTION OF THE LAPLACIAN IN $\mathbb{R}^2$

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1. Introduction

A conjecture of L. Payne [8] states that any second eigenfunction of the Laplacian with zero boundary condition for a bounded domain $\Omega \subseteq \mathbb{R}^2$ does not have a closed nodal line. This is also asked by S.-T. Yau [10, Problem 78] for Ω a bounded convex domain in $\mathbb{R}^2$.

L. Payne [9] proved the conjecture provided the domain $\Omega \subseteq \mathbb{R}^2$ is symmetric with respect to one line and convex with respect to the direction vertical to this line. Also. C.-S. Lin [7] proved the conjecture provided the domain $\Omega \subseteq \mathbb{R}^2$ is smooth, convex, and invariant under a rotation with angle $2\pi p/q$, where p and q are positive integers. Recently D. Jerison [5] proved the conjecture for long thin convex sets. Without any assumption on the smoothness of Ω he showed that the nodal line has to intersect $\partial\Omega$ in exactly two points.

In this paper we prove the conjecture when Ω is a bounded convex domain in $\mathbb{R}^2$ with C^∞ boundary.

To fix the notation for a bounded domain $\Omega \subseteq \mathbb{R}^2$ with smooth boundary we let u_2 be a second eigenfunction of Ω, that is, u_2 is a solution of the Dirichlet problem

$$\begin{cases} \Delta u_2 + \lambda_2 u_2 = 0 & \text{in } \Omega, \\ u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta = \sum_{i=1}^2 (\partial^2/\partial x_i^2)$ and λ_2 is the second eigenvalue of Ω.

The nodal line N of u_2 is defined by

$$N = \overline{\{x \in \Omega: u_2(x) = 0\}}. \tag{1.2}$$

The Courant nodal domain theorem implies that N must divide the domain Ω into exactly two components.

Our main result is the following:

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