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THE YAMABE PROBLEM ON MANIFOLDS WITH BOUNDARY

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A natural question in differential geometry is whether a given compact Riemannian manifold with boundary is necessarily conformally equivalent to one of constant scalar curvature, where the boundary is minimal. When the boundary is empty this is called the Yamabe Problem—so-called because, in 1960, Yamabe claimed to have solved this problem. In 1968, N. Trudinger found a mistake in Yamabe's paper [16] and corrected Yamabe's proof for the case in which the scalar curvature is nonpositive. In 1976, Aubin [1] showed that, if $\dim M \geq 6$ and M is not conformally flat, then M can be conformally changed to constant scalar curvature. In 1984, Richard Schoen [10] solved the Yamabe problem in the remaining cases.

In this paper, we study the problem in the context of manifolds with boundary and give an affirmative solution to the question formulated above in almost every case. In fact, we show that any compact Riemannian manifold with boundary and dimension 3, 4, or 5 is conformally equivalent to one of constant scalar curvature, where the boundary is minimal. When $n \geq 3$ and there exists a nonumbilic point at ∂M, the boundary of M, we show that the problem above has an affirmative answer. The remaining case is when $n \geq 6$ and ∂M is umbilic. Under these conditions we show that the problem above is solvable in the affirmative if either M is locally conformally flat, or the Weyl tensor does not vanish identically on the boundary.

The only case we do not consider in this paper is when $n \geq 6$, M is not locally conformally flat, ∂M is umbilic, and the Weyl tensor vanishes identically on ∂M. As a consequence of the above results we have the following theorem.

Theorem. *Any bounded domain in a Euclidean m-space* $\mathbb{R}^n$, *with smooth boundary and* $n \geq 3$, *admits a metric conformal to the Euclidean metric having constant scalar curvature and minimal boundary.*

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