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THE PICARD GROUP OF THE UNIVERSAL PICARD VARIETIES OVER THE MODULI SPACE OF CURVES

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1. Introduction

We denote by $\mathscr{M}_g^0$ the moduli space of smooth curves of genus g ($g \geq 3$) without automorphisms, and by $\mathscr{C}_g \xrightarrow{\pi} \mathscr{M}_g^0$ the universal curve over $\mathscr{M}_g^0$. For any integer d, we denote by $\psi_d \colon \mathscr{T}_g^d \to \mathscr{M}_g^0$ the universal Picard (Jacobian) variety of degree d; the fiber $J^d(C)$ over a point $[C]$ of $\mathscr{M}_g^0$ parametrizes line bundles on C of degree d, modulo isomorphism. The construction of these bundles can be found for example in [9]. Note that although for a fixed curve C the varieties $J^d(C)$ are all isomorphic to the Jacobian variety of the curve, it is not true that this isomorphism can be carried out over $\mathscr{M}_g^0$: For $d_1 \neq d_2$ the isomorphism $J^{d_1}(C) \cong J^{d_2}(C)$ depends on the choice of a line bundle on C of degree $d_1 - d_2$; on the other hand, except in the case where $d_1 - d_2$ is a multiple of $2g-2$, there is no "uniform" choice of a line bundle of degree $d_1 - d_2$ on the fibers of the universal curve (see Theorem 2). In this work we describe the Picard group of the $\mathscr{T}_g^d$'s; first a definition.

Definition. We define the relative Picard group of $\mathscr{T}_g^d$, denoted by $\mathscr{R}\operatorname{Pic}(\mathscr{T}_g^d)$, to be the cokernel of the map $\psi_d^* \colon \operatorname{Pic}(\mathscr{M}_g^0) \to \operatorname{Pic}(\mathscr{T}_g^d)$.

Lemma 1. *Two line bundles on $\mathscr{T}_g^d$ define the same element in $\mathscr{R}\operatorname{Pic}(\mathscr{T}_g^d)$ if and only if their restrictions to the fibers of the map ψ_d define isomorphic line bundles.*

Proof. This is a restatement of the see-saw principle (see [10]). q.e.d.

Since the Picard group of $\mathscr{M}_g^0$ is known (see [1]), we are going to describe the groups $\mathscr{R}\operatorname{Pic}(\mathscr{T}_g^d)$. As the first step for this, we shall describe a "weaker" group $\mathscr{N}(\mathscr{T}_g^d)$ (which we call the relative Neron-Severi group of $\mathscr{T}_g^d$) defined to be the group of line bundles on $\mathscr{T}_g^d$, modulo the rela-

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