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NOTE ON THE PERIODIC POINTS OF THE BILLIARD

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Marek Rychlik [2, Theorem 1.1] proves that for any bounded convex domain Ω in a Euclidean plane $\mathbf{R}^2$ with smooth *boundary* $X = \partial\Omega$ the set Fix_3 of all periodic points of period 3 of the billiard ball map related to Ω has empty interior in its (two-dimensional) phase space M_Ω. The last part of the proof of this theorem, considered in [2], involves a symbolic computation system. In this note a short elementary argument is presented which completes the proof in [2] without use of any computer programs. Combining this argument with §3 in [2], one gets also a direct proof of Theorem 1.2 of [2]: Fix_3 has Lebesgue measure zero.

We use the notation from [2], and state the results of [2] in a little more general form.

Theorem. *Let Ω be a bounded (note necessarily convex) domain in $\mathbf{R}^2$ with C^3-smooth boundary X. Then Fix_3 has empty interior and Lebesgue measure zero in M_Ω.*

Proof. Let $y_1, \cdots, y_n$ be the successive (transversal) reflection points of a periodic billiard trajectory in Ω. Consider a natural parametrization $h_i(x_i)$, $x_i \in \mathbf{R}$, of X around y_i with $\|h_i'(x_i)\| \equiv 1$, $\cos\varphi_i = \langle e_i, \nu_i\rangle > 0$, where $\nu_i = \nu(x_i)$ is the *unit normal* to X at $h_i(x_i)$, pointing into Ω, $\langle\cdot, \cdot\rangle$ is the natural inner product in $\mathbf{R}^2$, φ_i is the angle between e_i and ν_i, $0 < \varphi_i < \pi/2$, and

$$e_i = \frac{h_{i+1}(x_{i+1}) - h_i(x_i)}{\|h_{i+1}(x_{i+1}) - h_i(x_i)\|}.$$

One can introduce Φ_i and $\widehat{\Phi}_i$ simply by setting $\Phi_i = \cos\varphi_i$ and $\widehat{\Phi}_i = \sin\varphi_i$. Then, if $h_i(x_i)$ are the reflection points of a periodic trajectory, a simple computation gives

$$\frac{\partial l(x_i, x_{i+1})}{\partial x_i} = -\langle e_i, h_i'\rangle = -\cos\varphi_i = -\Phi_i = -\frac{\partial l(x_{i-1}, x_i)}{\partial x_i}.$$

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