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THE EXISTENCE OF NONMINIMAL SOLUTIONS TO THE YANG-MILLS EQUATION WITH GROUP SU(2) ON $S^2 \times S^2$ AND $S^1 \times S^3$

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Abstract

By generalizing Taubes' approach in [19], we construct an infinite number of gauge inequivalent irreducible SU(2)-connections over $S^2 \times S^2$ and $S^1 \times S^3$, which are nonminimal solutions to the Yang-Mills equations. These connections have a uniform background curvature, with concentrations near points, spaced evenly along a geodesic. Near half of these points the solution looks self-dual, and near the other half it looks anti-self-dual.

1. Introduction

Consider the Yang-Mills equations on a compact, oriented 4-dimensional Riemannian manifold M as the variational equations of a functional YM. The function space $\mathscr{B}$ is the space of isomorphism classes of pairs (P, A), where P is a principal G-bundle, $P \to M$, and A is a smooth connection on P. With respect to the C^∞-topology, $\mathscr{B} = \bigcup_n \mathscr{B}_n$ is the disjoint union of the spaces $\mathscr{B}_n$ which are indexed by $n \in \mathscr{Z}$. The integer n is minus the second Chern number $P \times_{\mathrm{SU}(2)} \mathscr{C}^2$. (This is the physicist's instanton number.)

Having fixed the Riemannian metric on the tangent space TM, the Yang-Mills functional is a natural, nonnegative functional on $\mathscr{B}$; this is an energy functional which measures the amount that a given connection's horizontal subbundle in TP fails to be involutive. It assigns to an orbit $[A] \in \mathscr{B}$ of a connection A the number

$$\mathrm{YM}(A) = \tfrac{1}{2} \int_M |F_A|^2 \, dv . \tag{1.1}$$

Here F_A is the curvature of the connection A, a section over M of the vector bundle $\Omega^2(\mathrm{Ad}\, P) = \mathrm{Ad}\, P \otimes \bigwedge^2 T^* M$, and $\mathrm{Ad}\, P$ is the associated vector bundle, $\mathrm{Ad}\, P = P \times_{\mathrm{Ad}} L(G)$ ($L(G)$ is the Lie algebra of G).

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