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DETECTING UNKNOTTED GRAPHS IN 3-SPACE

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Introduction

Definition. A finite graph Γ is *abstractly planar* if it is homeomorphic to a graph lying in S^2. A finite graph Γ imbedded in S^3 is *planar* if Γ lies on an embedded surface in S^3 which is homeomorphic to S^2.

In this paper we give necessary and sufficient conditions for a finite graph Γ in S^3 to be planar. (All imbeddings will be tame, e.g., PL or smooth.) This can be viewed as an unknotting theorem in the spirit of Papakyriakopolous [12]: a simple closed curve in S^3 is unknotted if and only if its complement has free fundamental group.

[12] can be viewed as a solution for Γ having one vertex and one edge. In [6] or [3, §2.3] this is extended: a figure-eight (bouquet of two circles) in S^3 is planar if and only if its complement has free fundamental group and each circle is unknotted. Gordon [4] generalizes this to all graphs with a single vertex: a bouquet of circles Γ in S^3 is planar if and only if its complement and that of any subgraph of Γ has free fundamental group. If fact, Gordon shows that this generalization of [6] is a fairly direct consequence of Jaco's handle addition lemma [8]. Far more difficult is Gordon's extension to the case in which Γ has two vertices, and no loops. We will require only the solution of the one-vertex case for our proof.

We will show:

7.5. Theorem. *A finite graph $\Gamma \subset S^3$ is planar if and only if*

(i) *Γ is abstractly planar,*

(ii) *every graph properly contained in Γ is planar, and*

(iii) *$\pi_1(S^3 - \Gamma)$ is free.*

There is an alternative formulation:

Theorem. *A finite graph $\Gamma \subset S^3$ is planar if and only if*

(a) *Γ is abstractly planar and*

(b) *for every subgraph $\Gamma' \subseteq \Gamma$, $\pi_1(S^3 - \Gamma')$ is free.*

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