J. DIFFERENTIAL GEOMETRY
34 (1991) 515-537

ALMOST RIEMANNIAN SPACES

CONRAD PLAUT

Introduction

We call a complete metric space (X, d) *almost Riemannian* if X is finite dimensional and d is a geodesically complete inner metric of (metric) curvature locally bounded below. This paper is an investigation of the local and global properties of these and more general inner metric spaces. Our two global results are generalizations of Toponogov's Comparison Theorem and Maximal Diameter Theorem. The latter is used to prove our main local result: that an almost Riemannian space is a topological manifold, and that its metric structure has an infinitesimal approximation by a Euclidean geometry (hence the name "almost Riemannian"). We also prove a precompactness theorem (cf. [6]) for any class of n-dimensional almost Riemannian spaces with fixed bounds on diameter and curvature.

In order to state these theorems precisely we need a few definitions (for more details see [17] and [18]). Throughout this paper X denotes a metrically complete inner metric space which is *convex* in the sense that every pair of points is jointed by a minimal curve. Convexity is implied by local compactness (and metric completeness). S_k will denote the simply connected, two-dimensional space form of curvature k. By *monotonicity* we mean the well-known fact that the angle between two minimal curves of fixed length in S_k is a monotone increasing function of the distance between the endpoints opposite the angle. A *geodesic terminal* is a point in X beyond which some geodesic cannot be extended. An open subset U of X is *geodesically complete* if it has no geodesic terminals.

Definition A. An open set U in X is said to be a *region of curvature* $\geq k$ if for every triangle $(\gamma_{ab}, \gamma_{bc}, \gamma_{ca})$ of minimal curves in U,

(a) there exists a representative $(\tilde{\gamma}_{AB}, \tilde{\gamma}_{BC}, \tilde{\gamma}_{CA})$ in S_k (i.e., $\tilde{\gamma}_{AB}, \tilde{\gamma}_{BC}, \tilde{\gamma}_{CA}$ are minimal of the same length as their correspondent curves) and

(b) for any y on γ_{AB} and Y on $\tilde{\gamma}_{AB}$ such that $d(y, a) = d(Y, A)$, we have $d(y, c) \geq d(Y, C)$.

Received April 13, 1990 and, in revised form, July 27, 1990.