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INSTABILITY OF THE CONSERVATIVE PROPERTY UNDER QUASI-ISOMETRIES

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Abstract

The paper shows by example the existence of a pair of quasi-isometrically equivalent complete Riemannian metrics g, $\tilde{g}$ on a manifold N so that (N, g) is conservative while $(N, \tilde{g})$ is not. This contrasts with known sufficient criteria for conservativeness which are stable under such changes in metric. In doing so we also give an example of a complete manifold with no nonconstant bounded solutions to Laplace's equation but with nonconstant bounded solutions to the heat equation.

1. Introduction

A smooth Riemannian manifold (N, g) has associated with it a number of standard and interrelated geometric objects. In particular, it possesses a unique geodesic through each point in each tangent direction, a Laplace Beltrami operator, and a Markovian family $(\mathbf{P}_n, n \in N)$ of measures on the continuous paths called Brownian motion. The measures $\mathbf{P}_n$ can be constructed as the limit where $\varepsilon \to 0$ of the simple random walk one obtains by: starting at n, using the metric to choose a direction uniformly at random, moving at speed $\frac{1}{\varepsilon}$ along the geodesic a distance $\sqrt{\varepsilon}$, and then repeating the process with this new point as starting point. The law of this process is a probability measure on paths in N; the weak limit $\mathbf{P}_n$ is Brownian motion. The semigroup P_t defined by $P_t f(n) = \mathbf{E}_n(f(X_t))$ has $\frac{1}{2}\Delta$ as the infinitesimal generator, where $\mathbf{E}_n$ is the expectation or integral against $\mathbf{P}_n$.

As the example of the usual disc $D = \{|z| < 1\} \subset \mathbf{C}$ shows, geodesics do not need to stay in N for all t and might explode (that is, eventually leave every compact set) in finite time. It is a well-known piece of analysis that any geodesic can be extended either until it explodes or for all time; oscilliatory discontinuities do not occur. If all geodesics extend for all

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