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CONVEX HYPERSURFACES WITH PRESCRIBED GAUSS-KRONECKER CURVATURE

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In [14, Problem 59] Yau raised the general question when a function F defined in a Euclidean 3-space $\mathbb{R}^3$ is the mean curvature or Gaussian curvature of a closed surface with prescribed genus. For mean curvature it was proposed to minimize the functional

$$\text{Area}(X) - \int_{\widetilde{X}} F$$

among all surfaces X of the same genus ($\widetilde{X}$ is the set bounded by X). However, it is not clear how the minimum, if it ever exists, should have the same genus. In this paper we study the latter problem for the Gauss-Kronecker curvature of a closed hypersurface in a Euclidean $(n+1)$-space $\mathbb{R}^{n+1}$ $(n \geq 1)$. If the given function is positive, the solution hypersurface is a priori convex. The difficulty of determining its topology is subdued and we can concentrate on the analysis.

Let X be a smooth hypersurface embedded in $\mathbb{R}^{n+1}$ and oriented with respect to its inner normal. Denote σ_k $(k = 0, 1, \cdots, n+1)$ the normalized kth elementary symmetric function of its principal curvatures. (Set $\sigma_0 = 1$ and $\sigma_{n+1} = 0$.) The following first variation formulas for the kth curvature integral, $k = 0, \cdots, n$, $I_k(X) = (n-k)^{-1} \int_X \sigma_k$, are valid [10]:

$$\delta I_k(X)\xi = \int_X \sigma_{k+1} \langle \xi, \nu \rangle. \tag{1}$$

Here ξ is a smooth variation vector field on X, ν is the unit outer normal, and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in $\mathbb{R}^{n+1}$. If we let $J_k(X) = I_k(X) - \int_{\widetilde{X}} F$, where F is a function defined in $\mathbb{R}^{n+1}$ and $\widetilde{X}$ is the compact subset bounded by X, we have

$$\delta J_k(X)\xi = \int_X (\sigma_{k+1} - F) \langle \xi, \nu \rangle.$$

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