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SUMS OF ELLIPTIC SURFACES

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Introduction

The classification of smooth, simply connected 4-manifolds remains one of the main unsolved problems of topology. A rich collection of examples is provided by the simply connected algebraic surfaces. A well-known folk conjecture states that perhaps all simply connected, smooth, closed 4-manifolds are connected sums of algebraic surfaces (possibly with some orientations reserved). This suggests the importance of studying the topology of algebraic surfaces—particularly, their behavior under connected sums and related constructions.

An important subclass of algebraic surfaces is comprised by the *elliptic surfaces*. These achieved fame in topology via Donaldson's invariants [4], [5], which showed that there are infinitely many diffeomorphism types of simply connected elliptic surfaces within each homeomorphism type [8], [9], [23], even though there is no classical smoothing uniqueness obstruction. In particular, these are infinite families of counterexamples to the smooth h-Cobordism conjecture for 4-manifolds. In some sense, "most" simply connected algebraic surfaces are elliptic. For any fixed integer $b \geq 10$, there are infinitely many diffeomorphism types of simply connected elliptic surfaces with $b_2 = b$, but only finitely many diffeomorphism types of simply connected, nonelliptic algebraic surfaces with $b_2 \leq b$.

Some progress has been made on the question of how algebraic surfaces behave under connected sum. Donaldson's invariants are "stable" under connected sum with $\overline{\mathbf{C}P^2}$, where the bar denotes reversed orientation. Thus, this operation (the algebraic geometer's "blowup") tends not to collapse diffeomorphism types. However, Mandelbaum [18], [20] and Moishezon [22] showed that for any M in a large class of algebraic surfaces (containing all simply connected elliptic surfaces and "complete intersections") the connected sum $M \sharp \mathbf{C}P^2$ (with the standard orientation

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