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GLOBAL ISOMETRIC EMBEDDING OF A RIEMANNIAN 2-MANIFOLD WITH NONNEGATIVE CURVATURE INTO A EUCLIDEAN 3-SPACE

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1. Introduction

The isometric embedding problem of a 2-dimensional Riemannian manifold M^2 with Gaussian curvature $K \geq 0$ into 3-dimensional Euclidean space $\mathbb{R}^3$ is one of many difficult problems. In fact, it is quite hard to show certain a priori estimates in a neighborhood of zero points of K and to verify the convergence of the Nash-Moser type interaction scheme, since the linearization operators are degenerating on $\{K = 0\}$. Lin [5] studied a local problem and solved it. Naturally, the next subject is a global problem, which we shall study in this paper. In a global case, Lin's method does not work well, though it is quite suggestive, since his technicalities are particularly adapted to the local situation. For instance, his ingenious parametrization would not lead to success in the global case. What we need are a new type of implicit function theorem and global a priori estimates for degenerating linearized operators.

Let $g = g_{ij}\, dx^i\, dx^j$ be a $C^{r,\alpha}$ Riemannian metric defined in $\mathbb{R}^2$, where $r \geq 2$ and $0 < \alpha < 1$ (actually, C^r smoothness will suffice for our purpose (cf. §5)). We assume that

$$(1.1) \qquad |g_{ij} - \delta_{ij}|_r \ll 1 \qquad (1 \leq i, j \leq 2),$$

where δ_{ij} stands for Kronecker's delta, $|\cdot|_r$ is the C^r supremum norm, and $A \ll 1$ means that A is sufficiently small. K denotes the Gaussian curvature of the Riemannian manifold $(\mathbb{R}^2, g)$. We assume

$$(1.2) \qquad K \geq 0,$$

and put $f = K \det(g_{ij})$. It is to be noted that (1.1) and (1.2) imply $0 \leq f \ll 1$. Let D be a bounded convex domain in $\mathbb{R}^2$ such that there exists a convex function $\phi \in C^\infty(\mathbb{R}^2)$ satisfying $\phi < 0$ in D and $\phi \geq 0$

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