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DIAMETER, VOLUME, AND TOPOLOGY FOR POSITIVE RICCI CURVATURE

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Dedicated to Wilhelm Klingenberg on the occasion of his 65th birthday

1. Introduction

A compact Riemannian n-manifold with (normed) Ricci curvature $\mathrm{ric} := \mathrm{Ric}/(n-1) \geq 1$ has diameter $\leq \pi$, and equality holds if and only if M is isometric to the unit n-sphere (Cheng's rigidity theorem, cf. [4], [12], [5]). The aim of the present paper is to prove the following theorem.

Theorem 1. *Let M^n be a compact Riemannian manifold with Ricci curvature ≥ 1. Let $-k^2$ be a lower bound of the sectional curvature of M^n, and ρ a lower bound of the injectivity radius. Then we may compute a number $\varepsilon = \varepsilon(n, \rho, k) > 0$ such that M is homeomorphic to the n-sphere whenever* $\mathrm{diam}(M) > \pi - \varepsilon$.

More precisely, $\varepsilon = v(\delta)/\mathrm{vol}(S^{n-1})$, where $v(r)$ denotes the volume of a ball of radius r in the unit n-sphere and

$$\delta = \begin{cases} \rho - \cosh^{-1}(\cosh(k\rho)^2)/(2k) & \text{for } k > 0, \\ (1 - \sqrt{2}/2)\rho & \text{for } k = 0. \end{cases}$$

For sectional curvature, a much stronger result is known:

Theorem 2 (Berger [3], Grove-Shiohama [8], [9]). *Let M^n be a compact Riemannian manifold with sectional curvature $K \geq 1$ and diameter $D > \pi/2$. Then M is homeomorphic to a sphere.*

One may not expect such a theorem for Ricci curvature since, e.g., for $M = S^m \times S^m$ with $\mathrm{ric} = 1$ we have $\mathrm{diam}(M) = (1 - 1/(2m-1))^{1/2} \cdot \pi$. So the bound on the diameter must depend at least on the dimension. A diameter pinching theorem for Ricci curvature in the diffeomorphism category was first stated by Brittain [2] (whose proof used an incorrect version of Gromov's compactness theorem) and proved by Katsuda [11, p. 13] using a result of Kasue [10]. However, the proof needs also an upper curvature bound, and it would be hard to compute the ε. We give a

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