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MOTION OF LEVEL SETS BY MEAN CURVATURE. I

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Abstract

We construct a unique weak solution of the nonlinear PDE which asserts each level set evolves in time according to its mean curvature. This weak solution allows us then to define for any compact set Γ_0 a unique generalized motion by mean curvature, existing for all time. We investigate the various geometric properties and pathologies of this evolution.

1. Introduction

We set forth in this paper rigorous justification of a new approach for defining and then investigating the evolution of a hypersurface in $\mathbb{R}^n$ moving according to its mean curvature. This problem has been long studied using parametric methods of differential geometry (see, for instance, Gage [15], [16], Gage-Hamilton [17], Grayson [19], Huisken [23], Ecker-Huisken [10], etc.). In this classical setup, we are given at time 0 a smooth hypersurface Γ_0 which is, say, the connected boundary of a bounded open subset of $\mathbb{R}^n$. As time progresses we allow the surface to evolve, by moving each point at a velocity equal to $(n-1)$ times the mean curvature vector at that point. Assuming this evolution is smooth, we define thereby for each $t>0$ a new hypersurface Γ_t. The primary problem is then to study geometric properties of $\{\Gamma_t\}_{t>0}$ in terms of Γ_0.

For the case $n=2$ this program has been successfully carried out in great detail (see [17], [19]). For $n \geq 3$, however, it is fairly clear that even if Γ_0 is smooth, a smooth evolution as envisioned above cannot exist beyond some initial time interval. Imagine, for instance, Γ_0 to be the boundary of a "dumbbell" shaped region in $\mathbb{R}^3$, as illustrated in Figure 1 (next page).

In view of Grayson [20] and numerical calculations of Sethian [35], we expect that as time evolves, the surface will smoothly evolve (and shrink)

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