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ON THE FORMATION OF SINGULARITIES IN THE CURVE SHORTENING FLOW

SIGURD ANGENENT

Abstract

In this paper the asymptotic behavior of solutions of the so called "curve shortening equation" for locally convex plane curves is studied. This is done by looking at the blow-up of solutions of the quasilinear parabolic PDE

$$\frac{\partial k}{\partial t} = k^2 \frac{\partial^2 k}{\partial \theta^2} + k^3$$

with periodic boundary conditions.

The main results (Theorems A, B, D, and D) may be summarized as follows: "small convex nooses get tightened by the Grim Reaper." In other words, a convex plane curve $C(t)$ $(0 \le t < T)$ which evolves according to its curvature will either shrink to a point in an asymptotically self similar manner (as described by Abresch & Langer, and Epstein & Weinstein), or else its maximal curvature will blow up faster than $(T-t)^{-1/2}$. In the second case, there is a sequence of times $t_n \uparrow T$ such that the curve obtained by magnifying $C(t_n)$ so that its maximal curvature becomes 1 will converge to the graph of $y = -\log\cos x$.

If the total curvature which disappears into the singularity is less than 2π, then it must actually be π. Moreover, the last statement of the previous paragraph is true for any sequence $t_n \uparrow T$, instead of just for some sequence. In this situation we also have an upper bound for the rate at which the maximal curvature $\kappa(t)$ of $C(t)$ blows up:

$$\kappa(t) \le \frac{C_\varepsilon}{(T-t)^{1/2+\varepsilon}}$$

for any $\varepsilon > 0$.

1. Introduction

In this paper we take a look at the way in which a plane immersed curve becomes singular, as it evolves according to its curvature.

Let S^1 be a unit circle, and $\mathbf{R}^2$ a Euclidean plane. A family of immersed curves $X: S^1 \times [0, T) \to \mathbf{R}^2$ evolves according to its curvature, if

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