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# ON TWISTOR SPACES OF THE CLASS $\mathscr{C}$

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## 0. Introduction

Let $M^{2n}$ be a $2n$-dimensional compact and connected oriented Riemannian manifold, and $Z(M)$ be its twistor space. The $M^{2n}$ for which $Z(M)$ is Kähler are classified, up to conformal equivalence, in [16], [13] for $n = 2$, in [24] for $n \geq 4$ and even, and in [3] for $n \geq 3$. The proofs are mainly differential-geometric.

Y. S. Poon has, however, constructed self-dual metrics on $\mathbb{P}_2(\mathbb{C}) \neq \mathbb{P}_2(\mathbb{C}) = M^4$ for which $Z(M)$ is in Fujiki's class $\mathscr{C}$ (i.e., bimeromorphic to a compact Kähler manifold), but *not* Kähler.

We show here that:

(1) for $n \geq 3$, $Z(M)$ is in $\mathscr{C}$ iff it is Kähler, iff $M^{2n} = S^{2n}$;

(2) for $n = 2$, if $Z(M)$ is in $\mathscr{C}$, then $M$ is either $S^4$, or *homeomorphic* to the connected sum of $\tau(M) > 0$ copies of $\mathbb{P}_2(\mathbb{C})$.

Apart from well-known facts, the proof consists in showing that if $Z(M)$ is in $\mathscr{C}$, then $\pi_1(M) = \pi_1(Z(M)) = 0$ where $\pi_1$ denotes the fundamental group.

This last equality is obtained by purely complex-geometric methods, using the simple-connectedness of the twistor fibers, and the compactness of the Chow scheme of manifolds in $\mathscr{C}$. More precisely, it is possible (see Theorem 2.2) to evaluate $\pi_1(Z)$, for $Z$ in $\mathscr{C}$, from $\pi_1(Y)$ and $\pi_1(A)$ if $A$ and $Y$ are compact connected submanifolds of $Z$, such that $Y$ has enough "deformations" meeting $A$ in $Z$. When $Y$ is a smooth rational curve with ample normal bundle in $Z$ (for example, a twistor fiber in $Z(M^4)$), and $A$ is a point on $Y$, we get, in particular, $\pi_1(Z) = 0$. This extends a former result of J. P. Serre on the fundamental group of a unirational variety.

## 1. Preliminaries

**1.1 Notation.** Let $X$ be any irreducible complex analytic space. Then $\pi_1(X) := \pi_1(X, a)$ for some unspecified $a$ in $X$.

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