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RIGIDITY AND THE DISTANCE BETWEEN BOUNDARY POINTS

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Introduction

In this paper we consider some rigidity problems in Riemannian geometry. In particular, we prove

Theorem A. *Any complete Riemannian metric without conjugate points on* $\mathbf{R}^n$ *which is isometric to the Euclidean metric outside a compact set must be isometric to the Euclidean metric.*

This was proved in the case $n = 2$ by Green-Gulliver [8] using E. Hopf's theorem [12] that any metric without conjugate points on a 2-torus must be flat. The corresponding question about n-tori is called the E. Hopf conjecture and is still open.

A corresponding theorem is true for hemispheres of the round Euclidean sphere:

Theorem B. *Any Riemannian metric on the open n-ball which has no conjugate points, and for which the complement of a compact set is isometric to the complement of a compact set in an open Euclidean hemisphere, must be isometric to the Euclidean hemisphere.*

The proofs of both these theorems rely on results that come from the following "boundary rigidity" problem: Let (M, H, g) and (M_1, H, g_1) be compact Riemannian manifolds with the same (i.e., diffeomorphic) smooth boundary H. The metric g on M induces a distance function d from $H \times H$ to $\mathbf{R}$, i.e., $d(h_1, h_2)$ is the distance in M between h_1 and h_2. For what (M, H, g) is it true that any (M_1, H, g_1) with $d = d_1$ must have g isometric to g_1? Such an (M, H, g) will be called boundary rigid.

This problem was considered previously by R. Michel [14] and M. Gromov [9]. They have shown that any compact subdomain of $\mathbf{R}^n$, any compact subdomain of an open n-dimensional hemisphere, and any compact subdomain of the hyperbolic plane are boundary rigid (see [9, §5.5B]). In

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