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AN ESTIMATE FOR HEXAGONAL CIRCLE PACKINGS

ZHENG-XU HE

1. Introduction

Let P be a circle packing in the complex plane $\mathbf{C}$, i.e., a collection of circles in $\mathbf{C}$ with disjoint interiors, and let c_0 be a circle of P. Suppose that for some positive integer $n \geq 2$, the n generations P_n of P about c_0 (defined successively by $P_0 = \{c_0\}$, $P_k = \{c \in P; c \in P_{k-1}$ or c is tangent to some circle of $P_{k-1}\}$, $k \geq 1$) is combinatorially equivalent to the n generations H_n of a regular hexagonal circle packing about one of its circles. Then the ratio of radii of any two circles of P tangent to c_0 is bounded by $1 + s_n$, where $s_2, s_3, \ldots$ is some decreasing sequence of positive numbers. We will denote by s_n the smallest possible constant with this property. In [7], B. Rodin and D. Sullivan showed that any circle packing which is combinatorially equivalent to an infinite regular hexagonal circle packing is also regular hexagonal, and as a consequence, s_n converges to 0. They conjectured that $s_n \leq C/n$ for some constant C. In this paper, we will prove this conjecture. This estimate for s_n is best possible as (we will see later) $s_n \geq 4/n$.

One may use our result to estimate the rate of convergence of the circle packing solutions f_ε to the Riemann Mapping Theorem given in [7], where ε is the size of the preimage circles, and of the approximating solutions f_δ to the Beltrami equations constructed in [4]. This shows that these solutions are constructive. Moreover, for the circle packing solutions f_ε of [7], we may combine with [6, Theorems 5 and 8] to conclude that the rate of convergence on compact subsets is of order at most $\varepsilon^{\alpha/8}$ for any fixed $\alpha < 1$, and their derivatives converge in L^∞ on compact subsets.

The proof of $s_n \leq C/n$ will be given in §2 with the assistance of an area estimate on the union of the images of the interstices bounded by the circles of H_n under the Schottky group generated by inversions of the circles of H_n (Lemma 2.2). In §3 we will prove this estimate. The argument also leads to vanishing of the Lebesgue measure of the limit

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