J. DIFFERENTIAL GEOMETRY
33 (1991) 379-394

MANIFOLDS NEAR THE BOUNDARY OF EXISTENCE

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Our primary purpose is to study relationships between bounds on sectional curvature K_M, diameter d_M, and volume V_M, together with their effects on the topology of closed Riemannian n-manifolds M. For purposes of illustration, we normalize for the moment all manifolds so that $d_M = \pi$. Consequently, $\min K_M \leq 1$ by the Bonnet-Meyers theorem, and from the Rauch comparison theorem, $V_M \leq V(n, \min K_M, \pi)$, where $V(n, k, D)$ denotes the volume of a D-ball, $B_k^n(\overline{p}, D)$ in the n-dimensional simply connected manifold of constant curvature k. We may thus represent any closed Riemannian n-manifold as a point in the $(\min K_M, V_M)$-plane (Figure 1, next page).

Any manifold to the right of a vertical line has *a priori* bounded Betti-numbers as Gromov proved in [6]. Moreover, in regions bounded to the left by a vertical line and below by a horizontal line above the axis, only finitely many homotopy types occur [10]. In fact, at least when $n \neq 3, 4$, such regions contain at most finitely many diffeomorphism types [15]. Understanding convergence and limit spaces with respect to the Gromov-Hausdorff distance is an essential tool in the proof of the latter result.

On the basis of these results, it is natural to examine the topological properties of manifolds close to the boundary of existence. As already explained, the existence region is bounded above by the curve $V(n, \cdot, \pi)$, to the right by the line $\min K_M = 1$, and below by the $\min K_M$-axis. Only two points, $(1, V(n, 1, \pi))$ and $(1/4, V(n, 1/4, \pi))$, on this boundary are actually represented by manifolds, namely the sphere and the real projective space of constant curvature. Moreover, any manifold located strictly between the vertical lines through these two extremal points is homeomorphic to S^n [14].

Received August 14, 1989. The authors were supported in part by grants from the National Science Foundation.