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SOME SPACES OF HOLOMORPHIC MAPS TO COMPLEX GRASSMANN MANIFOLDS

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Abstract

In this paper we study the topology of the space of based holomorphic maps of degree $-k$ from the Riemann sphere to complex Grassmann manifolds, which we denote by $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$. We compute $H_*(\mathrm{Rat}_k(\mathbb{G}_{n,n+m}))$ for all k, n and m as well as the natural inclusion $i(k, n, m)_*: H_*(\mathrm{Rat}_k(\mathbb{G}_{n,n+m}))\longrightarrow H_*(\Omega^2_k(\mathbb{G}_{n,n+m}))$ induced by forgetting the complex structures. These results also give the geometry of the moduli spaces of observable and controllable solutions to the linear control equations.

1. Introduction

Let $S^2 = \mathbb{CP}(1)$ denote the Riemann sphere and $\mathbb{G}_{n,n+m}$ the Grassmannian of all complex n-dimensional planes through the origin in $\mathbb{C}^{n+m}$. Both spaces are naturally complex manifolds and have natural base points (∞ and $\mathbb{C}^n\times\vec{0}\subset\mathbb{C}^{n+m}$, respectively). Let $\mathrm{Rat}(\mathbb{G}_{n,n+m})$ denote the space of all based holomorphic maps from (S^2, ∞) to $(\mathbb{G}_{n,n+m}, \mathbb{C}^n\times\vec{0})$ with the compact open topology. It is well known that every such holomorphic map is rational; that is, it is given by a series of zeros, poles, and residues, hence the terminology "Rat". In addition, associated to each element $f\in\mathrm{Rat}(\mathbb{G}_{n,n+m})$ is an integer $c(f) = k$, the total Chern number, given by the topological degree of $f: S^2\longrightarrow\mathbb{G}_{n,n+m}$. Thus $\mathrm{Rat}(\mathbb{G}_{n,n+m})$ breaks into components and it is known [5] that each component $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ is a connected complex manifold of complex dimension $(n+m)k$.

By forgetting the complex structure one obtains based continuous maps from S^2 to $\mathbb{G}_{n,n+m}$; that is, elements in the two-fold loop space $\Omega^2(\mathbb{G}_{n,n+m})$ whose components are also indexed by the degree $c(f)$. We

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