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MULTIPLE INTERSECTIONS ON NEGATIVELY CURVED SURFACES

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1. Introduction

Let X be a compact surface with a Riemannian metric of negative curvature. An n-tuple point on X is a point through which a single geodesic passes at least n times in different directions. The geodesic is not required to be closed. Our purposes are to describe the loci of triple and quadruple points and to show that the *quadruple points are dense* on X and the tangents at quadruple points can be made near to four given directions. The proof of this last fact is based on the theory of Anosov flows [1].

There is an elementary, intuitive way of studying geodesic triple points. We present it in §2 and §3 for the case of constant curvature. The quadruple points for variable curvature are studied in §4.

2. Hyperbolic surfaces

Consider a surface X of constant curvature minus one. Its universal cover is the hyperbolic plane $\mathbf{H}^2$.

Three different lines in $\mathbf{H}^2$ are lifts of the same geodesic on X if and only if there exist deck transformations A and B, each mapping one of the pair of lines onto the third. Such lines will pass through a common point ψ, if and only if their projection on X has a triple point at the image of ψ; this corresponds to $A\psi$, $B\psi$, and ψ being colinear.

There is a single degenerate configuration which occurs if ψ lies on the axis of A, B or $A^{-1}B$. In this case ψ projects to a single intersection and the line in $\mathbf{H}^2$ projects to a closed geodesic.

It is therefore natural, given two hyperbolic isometries A and B, say noncommuting and without fixed points in $\mathbf{H}^2$, to study the locus $Z_{A,B}$ of points ψ for which $A(\psi)$, $B(\psi)$, and ψ are colinear. Upon introducing coordinates, $Z_{A,B}$ can be described as the solution set to a cubic

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