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SPECIAL METRICS AND STABILITY FOR HOLOMORPHIC BUNDLES WITH GLOBAL SECTIONS

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Abstract

In this paper we describe canonical metrics on holomorphic bundles in which there are global holomorphic sections. Such metrics are defined by a constraint on the curvature of the corresponding metric connection. The constraint is in the form of a P.D.E which looks like the Hermitian-Yang-Mills equation with an extra zeroth order term. We identify the necessary and sufficient condition for the existence of solutions to this equation. This condition is given in terms of the slopes of subsheaves of the bundle and defines a property similar to stability. We show that if a holomorphic bundle meets this stability-like criterion, then its Chern classes are constrained by an inequality similar to the Bogomolov-Gieseker inequality for stable bundles.

0. Introduction

A fundamental feature of the geometry of holomorphic vector bundles is the existence of so-called *hermitian* or *metric* connections. These are the connections which are compatible both with the holomorphic structure of the bundle and with a hermitian bundle metric. In fact, given a hermitian bundle metric on a holomorphic bundle, the metric connection is uniquely determined. The correspondence thus obtained between metrics and connections is somewhat analogous to the relationship between riemannian metrics on smooth manifolds and the Levi-Civita connections on their tangent bundles. This relationship allows one (for example in the Yamabe problem) to define special riemannian metrics in terms of constraints on the curvatures of the corresponding Levi-Civita connections. The same can be done for holomorphic vector bundles; the Hermitian-Einstein (or Hermitian-Yang-Mills) equation [11], [12] can be thought of as a constraint on metrics in this way. It provides a criterion formulated in terms of curvatures whereby the holomorphic structure of a complex vector bundle determines a preferred hermitian bundle metric.

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