J. DIFFERENTIAL GEOMETRY
33 (1991) 139-168

COMPLETE SURFACES WITH FINITE TOTAL CURVATURE

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0. Introduction

The goal of this project is to verify a conjecture of Yau and the authors stated in [12] for dimension 2. The conjecture asserts that:

Conjecture. *Let M be an n-dimensional complete Riemannian manifold with nonnegative Ricci curvature. Assume that there exists a point $p \in M$ such that the volume of geodesic balls $B_p(r)$ centered at p with radius r satisfies*

$$\operatorname{Vol}(B_p(r)) = O(r^{\alpha}), \tag{0.1}$$

as $r \to \infty$ for some integer $\alpha \geq 1$. Let k be a nonnegative integer and $r(x)$ be the distance from p to x, and define

$$H_k(M) = \{f \mid \Delta f \equiv 0 \text{ and } |f|(x) = O(r^k(x))\}$$

to be the space of harmonic functions on M which do not grow faster than $r^k(x)$. Then the dimension of $H_k(M)$ must be at most the dimension of that in $\mathbf{R}^{\alpha}$, i.e.,

$$\dim(H_k(M)) \leq \dim(H_k(\mathbf{R}^{\alpha})). \tag{0.2}$$

Yau originally conjectured that $H_k(M)$ must be of finite dimension and its dimension is bounded by $\dim(H_k(\mathbf{R}^n))$, where $n = \dim M$. In 1989, the authors proved [12] that $H_1(M)$ has an estimate of the form $\dim(H_1(M)) \leq \alpha + 1$, where α is defined by (0.1). This lead us to the refinement of Yau's conjecture in the above form.

In this work, we will verify the conjecture (Theorem 4.6) for 2-dimensional manifolds with nonnegative curvature. In fact, it turns out that if we only assume the negative part of the Gaussian curvature is integrable, then there are rigid and powerful geometric and analytic consequences which are special because of the fact that we are dealing with surfaces. We

Received August 2, 1988 and, in revised form July 10, 1989. The first author's research was partially supported by an NSF grant, and the second author's research by the Earmarked Grant for Research, Hong Kong.