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THE VIRTUAL SOLVABILITY OF THE FUNDAMENTAL GROUP OF A GENERALIZED LORENTZ SPACE FORM

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Introduction

Let $\mathrm{Aff}_n(\mathbb{R})$ denote the group of all affine transformations of the real affine vector space $\mathbb{R}^n$. It is well known that $\mathrm{Aff}_n(\mathbb{R})$ is isomorphic to the semidirect product $\mathrm{Gl}_n(\mathbb{R}) \ltimes \mathbb{R}^n$, where $\mathbb{R}^n$ is identified with the group of all translations of $\mathbb{R}^n$. Let $\pi\colon \mathrm{Aff}_n(\mathbb{R}) \to \mathrm{Gl}_n(\mathbb{R})$ be the natural projection. A subgroup $\Gamma \subset \mathrm{Aff}_n(\mathbb{R})$ is called G-linear if $\pi(\Gamma) \subset G$, where G is a real algebraic group, i.e., G is the group $\mathbf{G}(\mathbb{R})$ of $\mathbb{R}$-points of an algebraic subgroup $\mathbf{G}$ of $\mathrm{Gl}_n(\mathbb{C})$ defined over $\mathbb{R}$. Let G^0 be the connected component of G, and let $G^0 = SR$ be the Levi decomposition of G^0, where R is the solvable radical of G, and S is a maximal semisimple subgroup of G^0. Let $S = S_1 S_2 \cdots S_r$ be an almost direct product of simple Lie subgroups S_i. The group Γ is called a group of generalized Lorentz motions if every S_i is a group of (real) rank $\mathrm{rk}_{\mathbb{R}} S_i \leq 1$. (By a rank of S_i we mean the dimension of any maximal $\mathbb{R}$-split torus in the Zariski closure $\mathbf{S}_i$ of S_i in $\mathbf{G}$.) Assume that Γ acts properly discontinuously on $\mathbb{R}^n$ (i.e., the set $\{\gamma \in \Gamma | \gamma K \cap K \neq 0\}$ is finite for every compact $K \subset \mathbb{R}^n$), and that the quotient $\mathbb{R}^n/\Gamma$ is compact. In the case where Γ is a group of Lorentz motions (that is $G = \mathrm{SO}(n-1, 1)$) it was proved in [9] that Γ is a virtually solvable group, i.e., Γ contains a solvable subgroup of finite index. The aim of the present paper is to prove similar results for all groups Γ of generalized Lorentz motions.

Theorem A. *Let Γ be a G-linear subgroup of* $\mathrm{Aff}_n(\mathbb{R})$. *Assume that* (a) *$\Gamma$ acts properly discontinuously on $\mathbb{R}^n$*, (b) *$\mathbb{R}^n/\Gamma$ is compact, and* (c) *Γ is a group of generalized Lorentz motions. Then Γ is a virtually solvable group.*

According to a result of G. A. Margulis [15] if Γ is a group of generalized Lorentz motions which acts properly discontinuously on $\mathbb{R}^n$ but $\mathbb{R}^n/\Gamma$ is not compact, then Γ is not necessarily a virtually solvable group.

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