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L^2-INDEX THEOREMS ON CERTAIN COMPLETE MANIFOLDS

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1. Introduction

Consider a Riemannian manifold M, Hermitian vector bundles E and F over M, and a first order elliptic differential operator $D: C^\infty(E) \to C^\infty(F)$. Such operators arise naturally from the Riemannian structure like the Gauss-Bonnet and the signature operator; more generally, one can consider the Dirac operators in the sense of [10]. Being a differential operator, D has closed extensions $\overline{D}$ mapping the Hilbert space $\mathscr{D}(\overline{D})$ (with the graph norm) to $L^2(F)$. In particular, there is the closure $D_{\min}$ and the maximal extension $D_{\max} = (D'_{\min})^*$, where $D': C^\infty(F) \to C^\infty(E)$ is the formal adjoint. If M is complete, then $D_{\max} = D_{\min}$ for all Dirac operators. Moreover, if M is compact, then $D_{\max}$ is a Fredholm operator, and its index is given by the celebrated Atiyah-Singer index formula. In general, D may or may not have a Fredholm extension. In this work we deal with a class of operators which need not be Fredholm but have a finite L^2-index in the sense that $\ker D \cap L^2(E)$ and $\ker D' \cap L^2(F)$ both have finite dimension; then we define

$$L^2\text{-}\operatorname{ind} D := \dim\ker D \cap L^2(E) - \dim\ker D' \cap L(F). \tag{1.1}$$

We will also assume that M is complete and $D_{\max} = D_{\min}$. Then if $D_{\min}$ is Fredholm, we have $\operatorname{ind} D_{\min} = L^2\text{-}\operatorname{ind} D$, but our assumptions will not imply the Fredholm property. Note that if D has a finite L^2-index, then a closed extension $\overline{D}$ is Fredholm if and only if the essential spectrum $\sigma_e(\overline{D}^*\overline{D})$ of the self-adjoint operator $\overline{D}^*\overline{D}$ has a positive lower bound. Still, the situation which we treat should be regarded as a type I case in the sense of [13].

Our model case is a complete manifold with finitely many ends which are all warped products. It follows from simple examples that the L^2-cohomology for such manifolds can be infinite, so we need a condition on the warping function f (formula (2.14) below) which allows at most

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