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INSTANTONS AND THE GEOMETRY OF THE NILPOTENT VARIETY

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1. Introduction

Let $\mathfrak{g}$ be the Lie algebra of a compact, connected, semisimple Lie group G, and let $\varphi: \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathbf{R}$ be the function

$$\varphi(A_1, A_2, A_3) = \sum_1^3 \langle A_i, A_i \rangle + \langle A_1, [A_2, A_3] \rangle,$$

where $\langle\ ,\ \rangle$ is an Ad-invariant inner product. We are going to study the trajectories of the gradient flow of φ. It turns out that the space of bounded trajectories is closely related to the *nilpotent variety*:

$$\mathscr{N} = \{x \in \mathfrak{g}^c | \mathrm{ad}(x) \text{ is nilpotent}\}.$$

Here $\mathfrak{g}^c$ is the complex Lie algebra $\mathfrak{g} \otimes \mathbf{C}$. On the other hand φ exhibits symmetries which are not immediately visible in $\mathscr{N}$; for example, the obvious action of SO(3) on $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ leaves φ invariant. Exploiting this we obtain some new information (and some old information) about the geometry of the nilpotent variety, enough to give an explanation for Brieskorn's result [1] that $\mathscr{N}$ has a finite quotient singularity along the codimension-2 orbits. We discuss Brieskorn's result and its relationship to the SO(3) action in §3. As a spin-off we find that the icosahedral group (for example) occurs naturally as the intersection of two three-dimensional subgroups, copies of SO(3), inside the compact group of type E_8.

The results concerning the trajectories of φ are stated in §2 and proved in §§4–6. Some standard information about nilpotent elements in semisimple Lie algebras is summarized in an appendix.

2. The gradient flow

To motivate the results of this section, it will be helpful to give a geometric interpretation of φ. We identify $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ with the space of linear

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