J. DIFFERENTIAL GEOMETRY
32 (1990) 457–472

ALMOST EINSTEIN MANIFOLDS OF NEGATIVE RICCI CURVATURE

MAUNG MIN-OO

1. Introduction

For a Riemannian manifold (M^n, g), denote by K its sectional curvature, a function on the Grassmannian bundle G_2M of 2-planes in the tangent bundle TM, by ρ its Ricci curvature, regarded as a function on the sphere bundle SM of unit tangent vectors, and by R its scalar curvature, which is a function on M. We normalize our curvature functions so that the sphere S^n of radius 1 has $K = 1$, $\rho = n - 1$ and $R = n(n-1)$. If M is compact, we denote by d its diameter, and by V its volume, and we define $r = ⨍ R = \frac{1}{V} \int R$ to be the average scalar curvature. (M^n, g) is called *Einstein* if the Ricci curvature ρ is a constant function $= \frac{r}{n}$.

This paper is concerned with compact almost Einstein manifolds of negative average scalar curvature, where ρ is almost a constant and $r < 0$.

For $n \geq 3$ and $\Lambda > 0$, we define $\mathscr{M}^-(n, \Lambda)$ to be the set of all smooth compact Riemannian manifolds (M^n, g) of dimension n, satisfying the following curvature bounds:

(i) $r < 0$,
(ii) $d^2 \max |K| \leq \Lambda^2$.

It is well known [4] that if $n \geq 3$, then any smooth compact manifold M^n admits a metric g with $r < 0$.

The main result of this paper is the following pinching theorem for the Ricci curvature.

Theorem. *For any $n \geq 3$ and $\Lambda > 0$, there exists an $\varepsilon(n, \Lambda) > 0$, depending only on n and Λ, such that if $(M^n, g) \in \mathscr{M}^-(n, \Lambda)$ and if its Ricci curvature satisfies*

$$\max_{SM} |n\rho/r - 1| < \varepsilon(n, \Lambda),$$

then M admits an Einstein metric $\overline{g}$ with $\rho(\overline{g}) \equiv -1$.

Received December 17, 1987 and, in revised form, February 13, 1989. This work was partially supported by the National Sciences and Engineering Research Council of Canada.