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LIE'S THIRD THEOREM FOR INTRANSITIVE LIE EQUATIONS

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Introduction

In [4], H. Goldschmidt used the formalism developed by B. Malgrange [9] to prove Lie's third theorem in the context of transitive Lie algebras: "If $L_{k+1} \subset J_{k+1}T\mathbf{R}_0^m$, where $k > 0$, is a $(k+1)$-truncated transitive Lie algebra such that the symbol of $L_k = \pi_k L_{k+1}$ is 3-acyclic, then there exists a formally integrable analytic Lie equation $R_k \subset J_k T\mathbf{R}^m$ such that $R_{k+1,0} = L_{k+1}$."

In this paper, we show that the above R_k can be constructed without using the Cartan-Kähler theorem; our proof only requires Frobenius' theorem. Consequently, in the intransitive case, we are able to prove a version of E. Cartan's results [1] without assuming that the structure functions c_{ijk} and $a_{ij\lambda}$ are analytic.

Our main result is the following theorem, which we state here only in the transitive case for simplicity.

Theorem. *Suppose* $L_{k+2} \subset J_{k+2}T\mathbf{R}_0^m$, *where* $k > 0$, *is a* $(k+2)$-*truncated transitive Lie algebra. Then there exists a* C^∞ *vector sub-bundle* $R_{k+1} \subset J_{k+1}T\mathbf{R}^m$ *such that*:

(i) $R_k = \pi_k(R_{k+1})$ *is a vector sub-bundle of* $J_k T\mathbf{R}^m$;
(ii) $[R_{k+1}, R_{k+1}] \subset R_k$;
(iii) $R_{k+1,0} = L_{k+1}$;
(iv) $R_{k+1} \subset (R_k)_{+1}$

If the symbol of $L_k = \pi_k L_{k+1}$ is 3-acyclic, then L_{k+1} can be prolonged to L_{k+2}. We know that all its prolongations are isomorphic, thus the assumption in Goldschmidt's theorem gives us a $(k+2)$-truncated transitive Lie algebra.

The equation R_k in the Theorem may not be formally integrable (we only know that $\pi_k\colon (R_k)_{+1} \to R_k$ is surjective). However, when the symbol of L_k is 2-acyclic, Theorem 4.1 of Goldschmidt [2] implies that R_k

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