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UNIQUENESS OF THE COMPLEX STRUCTURE ON KÄHLER MANIFOLDS OF CERTAIN HOMOTOPY TYPES

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1. Introduction

In this note we show that the homotopy types of certain complex projective spaces and quadrics support a unique complex structure of Kähler type. Structures on complex projective space have attracted much attention. Hirzebruch and Kodaira [14], [11, p. 231] showed that a Kähler manifold V with the homotopy type and Pontryagin classes of CP_n is analytically equivalent to CP_n; their additional assumption that $c_1(V) \neq -(n-1)x$ for even n was later removed by Yau's work [31]. Here x denotes the generator of $H^2(V; Z)$ which is positive in the sense that it is the fundamental class of some Kähler metric on V [13, §18.1]. On the other hand it is known that for every $n > 2$ the homotopy type of CP_n supports infinitely many inequivalent differentiable structures distinguished by their Pontryagin classes (see Montgomery and Yang [25] or Wall [30] for $n = 3$ and Hsiang [15] for $n > 3$). Moreover for $n = 3$ or 4 each of these smooth structures can be shown to support almost complex structures. In §7 we prove this for the case $n = 4$ by applying results of Brumfiel and Heaps. The main result of this paper is that for $n \leq 6$ these other smoothings of a homotopy CP_n do not support a Kähler structure.

Theorem 1. *A Kähler manifold homotopy equivalent to CP_n for $n \leq 6$ is analytically equivalent to CP_n.*

It follows from the Kodaira embedding theorem that any homotopy complex projective space with a Kähler structure is projective algebraic, i.e., is analytically equivalent to a nonsingular subvariety of a higher dimensional projective space [13, §18.1]. In [31], Yau applied a criterion of Kodaira to show that a complex manifold homotopy equivalent to CP_2 is algebraic (hence Kähler) and showed moreover that it is analytically equivalent to CP_2. It is still an open question whether a complex manifold

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