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AUTOMORPHIC FORMS OF $\bar{\partial}$-COHOMOLOGY TYPE AS COHERENT COHOMOLOGY CLASSES

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The arithmetic theory of holomorphic automorphic forms is most naturally treated in terms of a certain family of vector bundles on Shimura varieties, called *automorphic vector bundles.* Let $M = \Gamma\backslash X$ be a connected component of a Shimura variety; here X is the Hermitian symmetric space of noncompact type associated to the semisimple Lie group G and Γ is an arithmetic subgroup of G. We write $X = G/K_\infty$; let σ be a finite-dimensional representation of K_∞; then σ determines an automorphic vector bundle $[\mathscr{V}_\sigma]$ over M (cf. §2 for definitions). We view $[\mathscr{V}_\sigma]$ as a locally free coherent sheaf over the quasiprojective variety M.

When M is compact, its cohomology with coefficients in $[\mathscr{V}_\sigma]$ can be computed by applying Hodge theory to the Dolbeault complex of $[\mathscr{V}_\sigma]$, endowed with a G-invariant metric. This situation has been studied by a number of authors, notably by Schmid [42]. The harmonic $(0, q)$-forms with values in $[\mathscr{V}_\sigma]$ correspond to the occurrence in $L_2(\Gamma\backslash G)$ of a certain class of unitary representations of G: namely, those with $\bar{\partial}$-cohomology with coefficients in σ, defined as in §4 below. Here the $\bar{\partial}$-cohomology of the representation π with coefficients in σ is defined to be the relative Lie algebra cohomology group $H^*(\mathfrak{P}, K_\infty, \pi_0 \otimes \sigma)$, where $\mathfrak{P}$ is a subalgebra of $\mathrm{Lie}(G)_\mathbb{C} = \mathfrak{g}_\mathbb{C}$ containing $\mathrm{Lie}(K_\infty)_\mathbb{C}$ and π_0 is the $(\mathfrak{g}, K_\infty)$ module associated to π.

The unitary representations with $\bar{\partial}$-cohomology have yet to be classified. They include the discrete series and, more generally, any representation of G with $(\mathfrak{g}, K)$-cohomology (in the sense of [12] and [53]), but there are others as well which play an important role in applications, as we explain below (cf. §4).

It is known [25] that automorphic vector bundles have models over number fields which are compatible with the canonical models [47], [34] of Shimura varieties; thus their sections rational over a given number field k define k-arithmetic (holomorphic) automorphic forms. Shimura has

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