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ESTIMATING $\|d\varphi^t\|$ FOR UNIT VECTOR FIELDS WHOSE ORBITS ARE GEODESICS

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Introduction

In the following, all manifolds, vector fields, etc., will be assumed to be real analytic. Let M be a connected, n-dimensional, complete riemannian manifold, and v a unit vector field (i.e., $|v| \equiv 1$) all of whose orbits are geodesics of M (i.e., $\nabla_v v \equiv 0$).

Although it is perhaps not really necessary, we also assume that *not all orbits of v are closed*, otherwise, by Wadsley's Theorem [11], there exists an S^1-action on M with the same orbits as v, and our problems should probably be studied in that context.

At each point $x \in M$ define $e_x = \max\{|\nabla_z v|^2 - K_{zv}\}$, where z ranges over all unit vectors $z \in T_x M$ perpendicular to v; here K_{zv} denotes the sectional curvature of M at x with respect to the 2-plane spanned by z and v.

Let φ^t be the flow generated by v and, for each $x \in M$ and any $t \geq 0$, define $E_{xt} = \max e_y$, where y ranges over the orbit interval $[\varphi^{-\sqrt{2}t}(x),\ \varphi^{\sqrt{2}t}(x)]$.

Theorem II. *Assume $x \in M$ is such that $e_x \geq 0$ (for example, at x suppose $K_{zv} \leq 0$ for some z as above). Then for any unit vector $u \in T_x M$ and all $t \geq 0$ we have*

$$|d\varphi^t(u)|^2 + |d\varphi^{-t}(u)|^2 \leq 2\cosh^2 t\sqrt{E_{xt}}.$$

Examples. (i) In the (trivial) case when v is also a Killing vector field, it is easy to see that $e_x \equiv 0$ and our inequality is sharp in this case (see §5).

(ii) If v is the geodesic flow on the unit sphere bundle SM^2 of a surface M^2, and we consider the curvature as a function $K: M \to R$, then at the point $x = (y, \xi_y)$ of SM^2

$$2e_x = (K-1)^2 + [(K^2-1)^2 + (dK(\xi))^2]^{1/2} \geq 0$$

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