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HARMONIC MAPS FROM A 2-TORUS TO THE 3-SPHERE

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0. Introduction

There have been many advances in recent years in the theory of harmonic maps of Riemann surfaces to spheres or symmetric spaces. These give constructions which produce harmonic maps from algebraic curves in associated complex manifolds (e.g. [7], [10]) and which provide many new examples, but unfortunately none of these methods says anything about the simplest and most basic situation of a harmonic map into the 2-sphere or the 3-sphere. Indeed, Bryant has shown that, however many derivatives one takes, there is no way of obtaining a construction like this for a minimal surface in the 3-sphere S^3, a particular case of a harmonic map.

In this paper we shall tackle this same simple basic situation, in the case where the Riemann surface is a torus with any conformal structure. We shall show that the equations for a harmonic map from a torus to S^3 reduce in a different way to algebraic geometry, in fact the geometry of a hyperelliptic curve, which we call the *spectral curve* Σ. This curve has finite genus and is constrained by integrality conditions on the periods of certain differentials of the second and third kind. These constraints are difficult to handle in general, but we shall show the existence of new examples of harmonic maps, and in particular minimal tori in S^3, by finding suitable curves. Furthermore, the method of solution shows that harmonic maps to S^3 are by no means rigid in general. They admit deformations which are parametrized by a real torus, of dimension p, where p is the genus of Σ.

This method of solution has its origins both in the theory of integrable systems like the KdV equation or sinh-Gordon equation and in the study of magnetic monopoles via the Bogomolny equations, where in both cases an algebraic curve lies at the heart of the solution. It is the analogy with

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