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CASSON'S INVARIANT AND GAUGE THEORY

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Recently, Andrew Casson [7] (see [2] or [16]) defined an integer valued invariant for closed 3-manifolds with the homology of S^3. Casson gave a topological definition of his invariant. An analytic definition of Casson's invariant is the subject of this article. Roughly speaking, Casson's invariant can be defined using gauge theory as an infinite dimensional generalization of the classical Euler characteristic.

The article begins below with an introduction to the relevant geometry of the space of connections on a homology 3-sphere. In §2, the definition of an integer valued invariant of an oriented homology 3-sphere is given. The construction of Casson's integer valued invariant is reviewed in §3 where the main theorem is stated: These two invariants are equal. §§4–8 are occupied with the proof of the main theorem. There is also a technical appendix.

1. Gauge theory in 3 dimensions

The new definition of Casson's invariant requires some basic facts from gauge theory (connections, curvature and covariant derivatives); the reader is referred to [11] and [15] for these definitions. Related material is present in the recent work of Andreas Floer [10].

Fix an oriented, closed 3-manifold, M, with the homology of S^3. Every principal SU(2) bundle over M is isomorphic to the trivial bundle, $P \equiv M \times \mathrm{SU}(2)$. It is convenient (though not necessary) to fix a trivialization of P and the associated product connection, Γ.

The space of smooth connections on P, $\mathscr{A} \equiv \mathscr{A}(P)$, is an affine space; the choice of Γ gives an affine isomorphism of $\mathscr{A}$ with $\Omega^1 \times \mathrm{su}(2)$. Here, su(2) is the Lie algebra of SU(2), and Ω^p $(p = 0, 1, 2, 3)$ is the space of smooth p-forms on M. Use the L^2_1-inner product on $\Omega^1 \times \mathrm{su}(2)$ to define

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